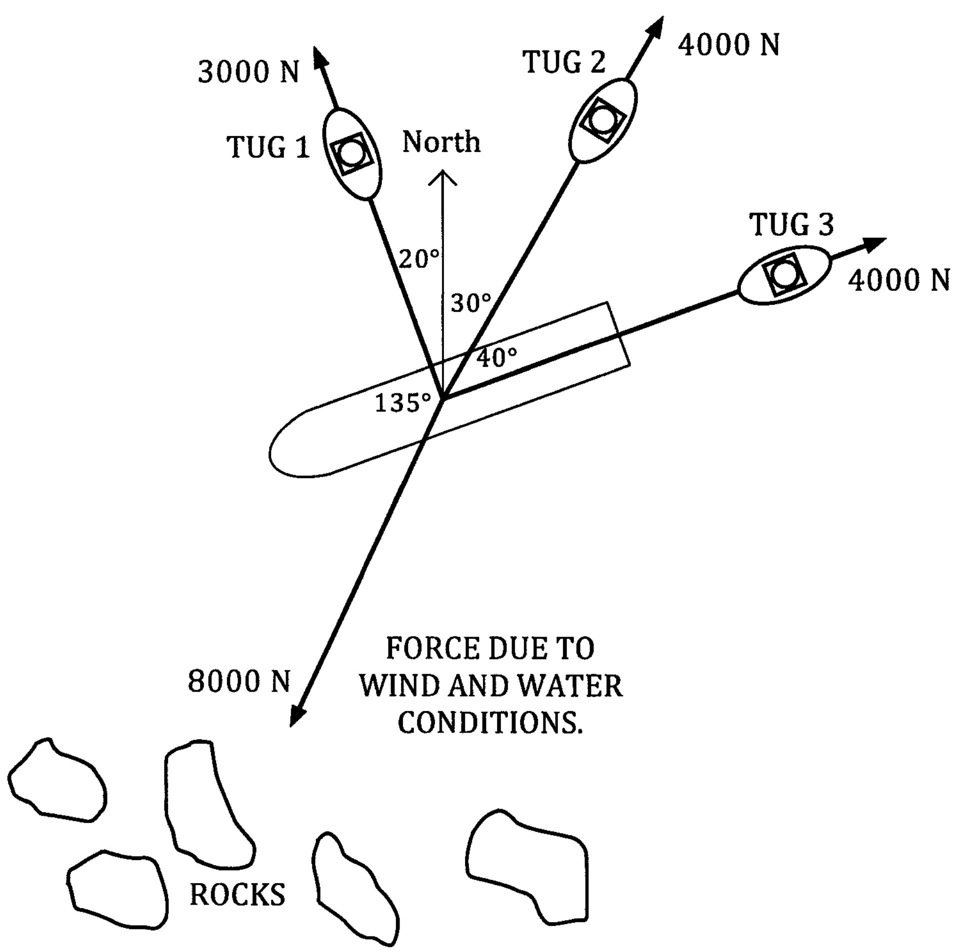
Chapter 4.

Vectors in component form.

Situation

A crippled oil tanker, with its engines out of action, is drifting towards some rocks. Tow lines have been attached to the tanker and three tugs are attempting to pull the boat away from the rocks. The forces exerted by the tugs and by the wind and water conditions are as shown below.



Find the magnitude and direction of the resultant force acting on the tanker.

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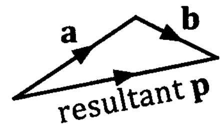
How did you get on with the situation on the previous page? It involved finding the resultant of a number of vectors. Did you sum the vectors using one of the methods from chapter 3 i.e. scale drawing or by using trigonometry?

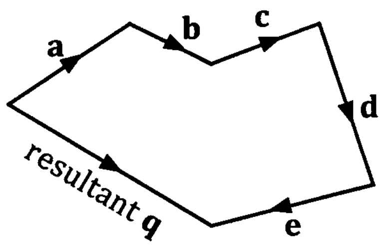
To use scale drawing we can extend the "nose to tail" idea of the triangle of vectors:

$$\mathbf{p} = \mathbf{a} + \mathbf{b}$$



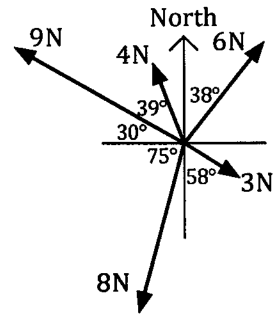
to a polygon of vectors:

$$\mathbf{q} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + \mathbf{e}$$

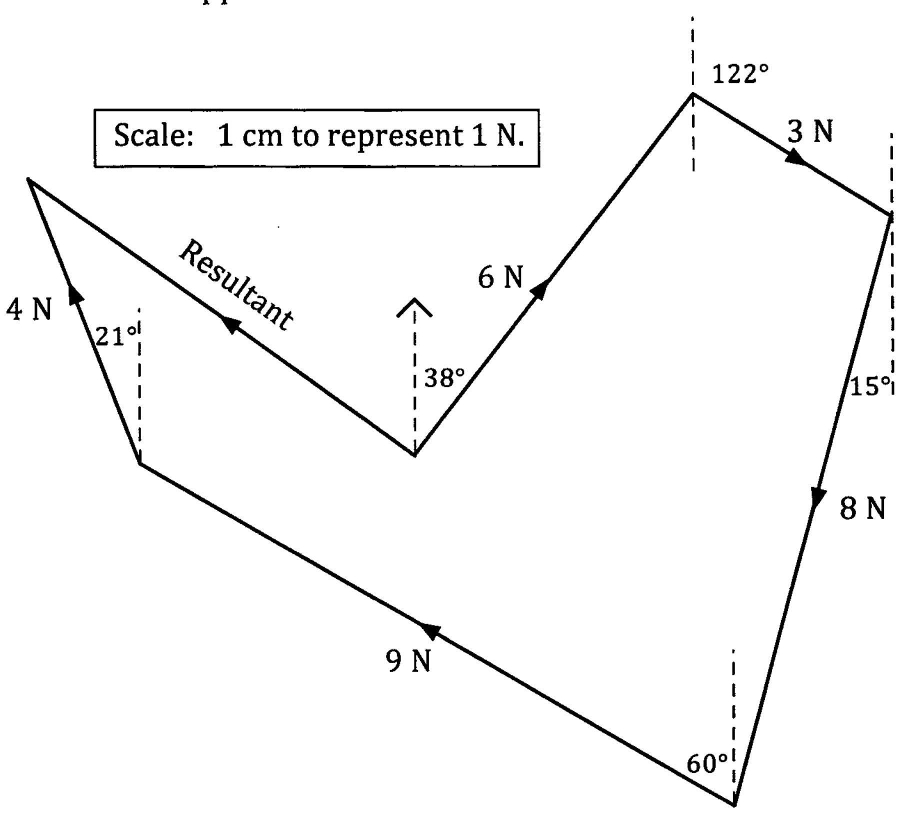


Example 1

A body experiences five forces as shown in the diagram on the right. Use a scale drawing to determine the magnitude and direction of the resultant of these forces.

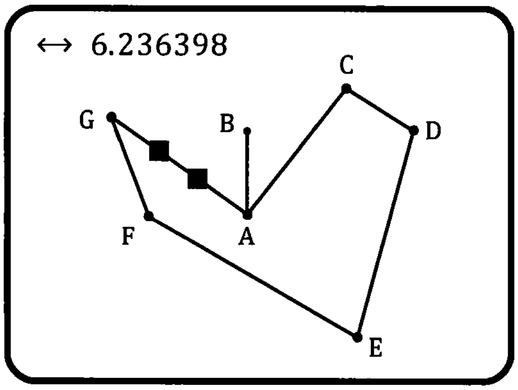


Choose a suitable scale and draw the forces "nose to tail". The resultant will be given by the line segment completing the polygon and in the opposite sense to the other forces:



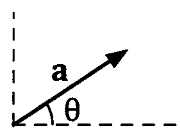
The resultant has a magnitude of ~6·2 N at ~306°.

Alternatively we could create the scale diagram using the drawing capability of some calculators.

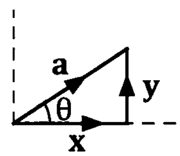


An alternative to using a scale drawing to determine the resultant of the five forces involves expressing each vector in terms of its **components** in two mutually perpendicular directions e.g. horizontal and vertical.

Consider the vector **a** shown on the right.



The vector triangle illustrated shows **a** expressed as the sum of the horizontal vector **x** and the vertical vector **y**.



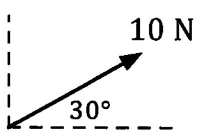
Thus $\mathbf{a} = \mathbf{x} + \mathbf{y}$

By trigonometry: $\cos\theta = \dfrac{|\mathbf{x}|}{|\mathbf{a}|}$ i.e. $|\mathbf{x}| = |\mathbf{a}|\cos\theta$

and $\sin\theta = \dfrac{|\mathbf{y}|}{|\mathbf{a}|}$ i.e. $|\mathbf{y}| = |\mathbf{a}|\sin\theta$

Thus **a** can be expressed as the sum of $|\mathbf{a}|\cos\theta$ units horizontally and $|\mathbf{a}|\sin\theta$ units vertically.

e.g.



The 10 N force shown in the diagram can be expressed as

$10\cos 30°$ N horizontally + $10\sin 30°$ N vertically

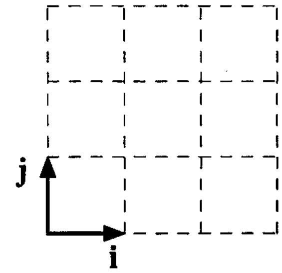
i.e. $5\sqrt{3}$ N horizontally + 5 N vertically.

However ① It is rather tedious to have to write "horizontally" and "vertically" every time.

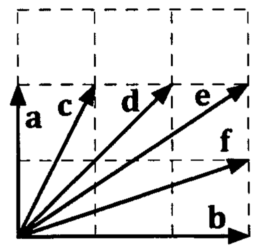
② Horizontally could mean to the left or to the right and vertically could mean up or down (at this stage we are only considering vectors in two dimensions so do not have to consider vectors coming out of the page).

See over the page for how we cope with these points. ⇨

To avoid these problems we use **i** and **j** to represent horizontal and vertical **unit vectors** (i.e. vectors of unit length) as shown in the diagram on the right.



The vectors **a**, **b**, **c**, **d**, **e** and **f** shown in the diagram can be expressed in terms of these unit vectors as follows.



$\mathbf{a} = 2\mathbf{j}$ $\qquad$ $\mathbf{b} = 3\mathbf{i}$

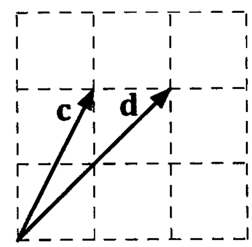
$\mathbf{c} = \mathbf{i} + 2\mathbf{j}$ $\qquad$ $\mathbf{d} = 2\mathbf{i} + 2\mathbf{j}$

$\mathbf{e} = 3\mathbf{i} + 2\mathbf{j}$ $\qquad$ $\mathbf{f} = 3\mathbf{i} + \mathbf{j}$

For the vector a**i** + b**j**, "a" is said to be the **horizontal component** and "b" the **vertical component**.

Once vectors are expressed in this a**i** + b**j** form it becomes easy to manipulate them.

For example consider again the vectors **c** and **d**.



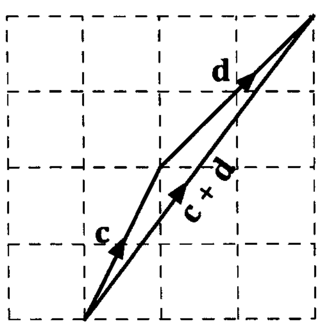
As before, $\mathbf{c} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{d} = 2\mathbf{i} + 2\mathbf{j}$.

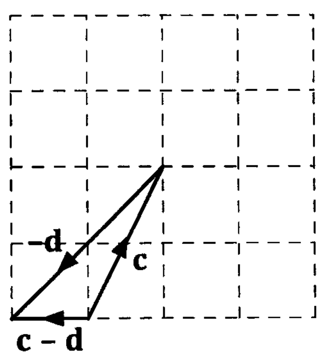
The working and diagrams below show **c** + **d**, **c** − **d** and 2**c**.

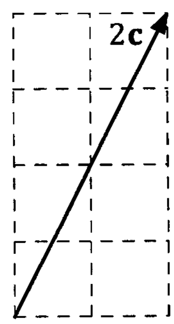
$$\mathbf{c} + \mathbf{d} = (\mathbf{i} + 2\mathbf{j}) + (2\mathbf{i} + 2\mathbf{j}) = 3\mathbf{i} + 4\mathbf{j}$$

$$\mathbf{c} - \mathbf{d} = (\mathbf{i} + 2\mathbf{j}) - (2\mathbf{i} + 2\mathbf{j}) = -\mathbf{i}$$

$$2\mathbf{c} = 2(\mathbf{i} + 2\mathbf{j}) = 2\mathbf{i} + 4\mathbf{j}$$

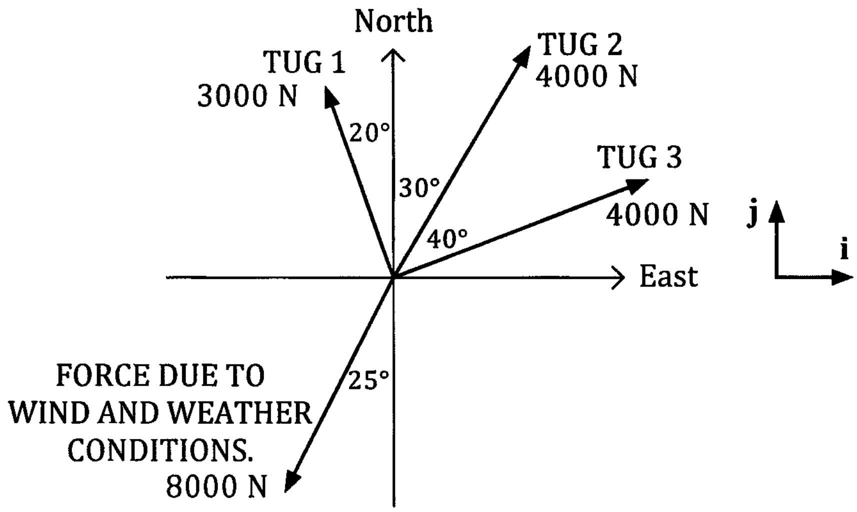






Now that we have seen how vectors can be expressed in the form **ai** + **bj** how can this be used to sum a number of vectors, as the situation at the beginning of this chapter required us to do?

We were required to sum four vectors as shown in the diagram on the right.



Expressing each in the form a**i** + b**j**:

Force from Tug 1:	(-3000 cos 70° **i** + 3000 sin 70° **j**) N
Force from Tug 2:	(4000 cos 60° **i** + 4000 sin 60° **j**) N
Force from Tug 3:	(4000 cos 20° **i** + 4000 sin 20° **j**) N
Force due to wind and water:	(-8000 cos 65° **i** - 8000 sin 65° **j**) N

Thus the resultant of these forces will be

$$[(-3000\cos 70° + 4000\cos 60° + 4000\cos 20° - 8000\cos 65°)\,\mathbf{i}$$
$$+\ (3000\sin 70° + 4000\sin 60° + 4000\sin 20° - 8000\sin 65°)\,\mathbf{j}\,]\text{ Newtons}$$
$$\approx\ (1352\mathbf{i} + 401\mathbf{j})\text{ Newtons}$$

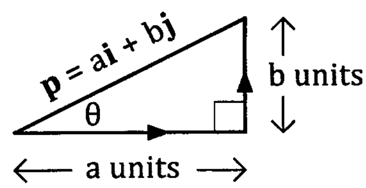
i.e. a vector of magnitude $\sqrt{1352^2 + 401^2} \approx 1410$ Newtons, in a direction ~073°.

Note:
- To find the magnitude and direction of the resultant from the component form we used Pythagoras and trigonometry.

  In the general case, if $\mathbf{p} = a\mathbf{i} + b\mathbf{j}$ then **magnitude**, or **modulus**, of **p** is given by

  $$|\mathbf{p}| = \sqrt{a^2 + b^2}$$

  and θ is found using $\tan\theta = \frac{b}{a}$.

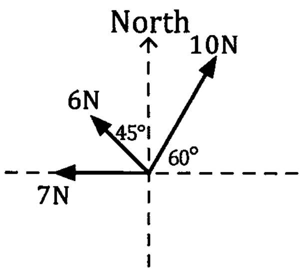
- The vector $p\mathbf{i} + q\mathbf{j}$ is sometimes written as an ordered pair (p, q), or perhaps $<p, q>$, and sometimes as a *column matrix* $\begin{pmatrix} p \\ q \end{pmatrix}$.

  Whilst the reader needs to be aware of these alternative ways of writing the vector $p\mathbf{i} + q\mathbf{j}$, at this stage this book will tend to use the $p\mathbf{i} + q\mathbf{j}$ form most frequently.
- Some calculators have built in routines for changing a vector given in component form to its magnitude and direction. Does your calculator have this ability?

Exercise 4A

For numbers 1 to 4 use a scale drawing to determine the magnitude and direction of the resultant of each system of vectors.

1.



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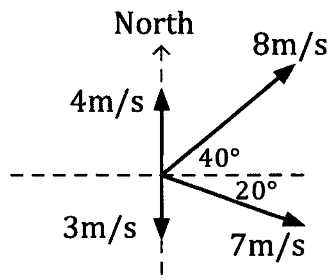

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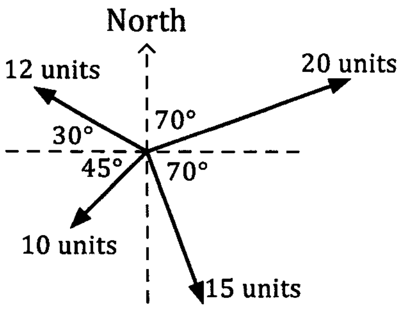

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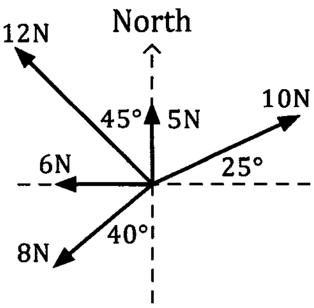
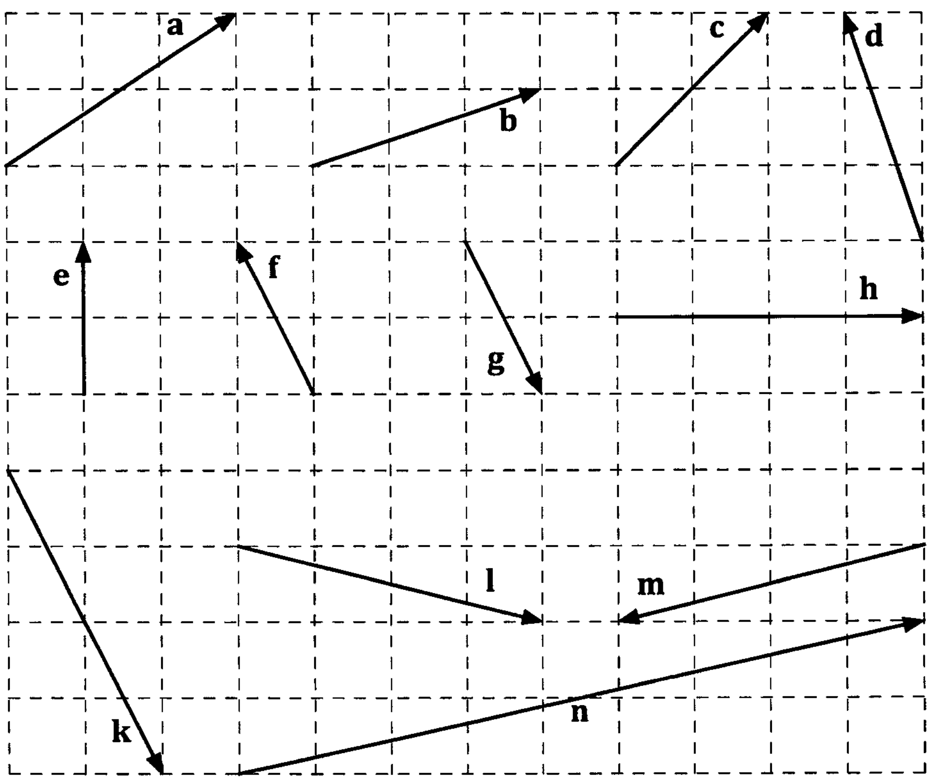


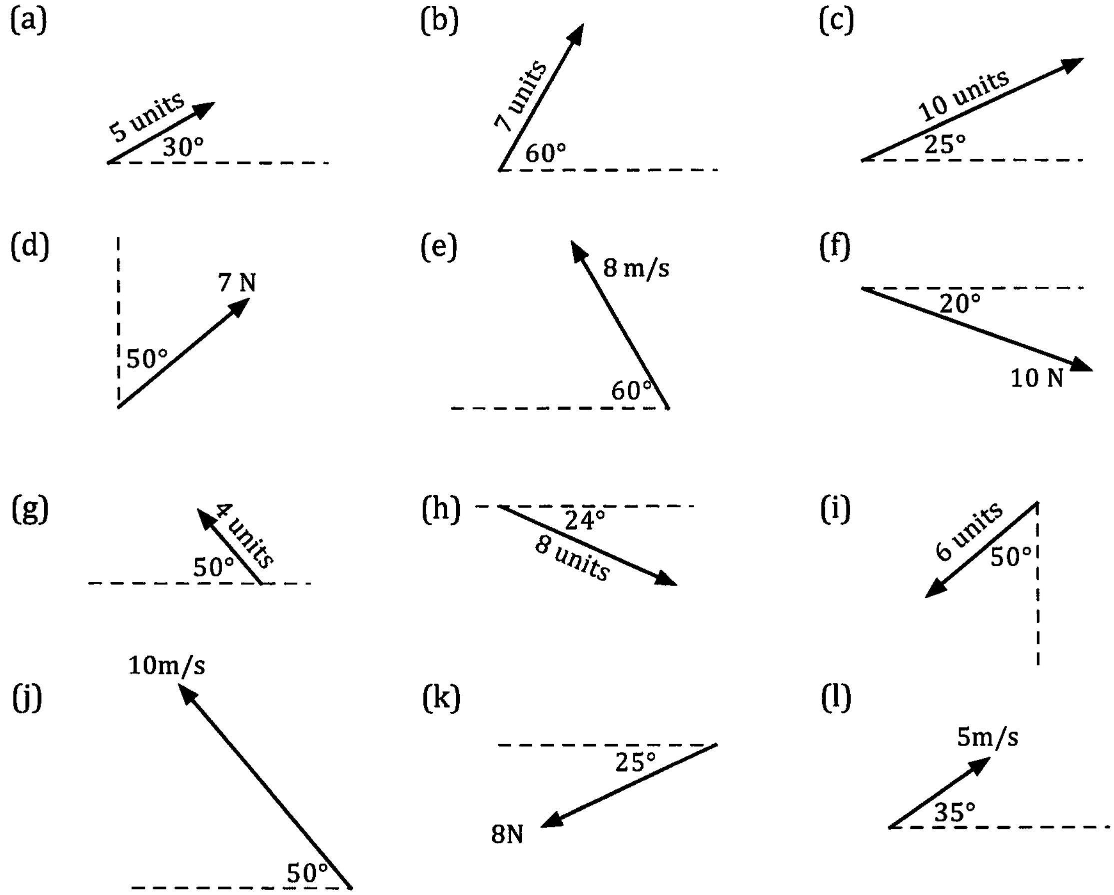
5. Express each of the vectors shown below in the form a**i** + b**j** where **i** is a unit vector to the right and **j** is a unit vector up.



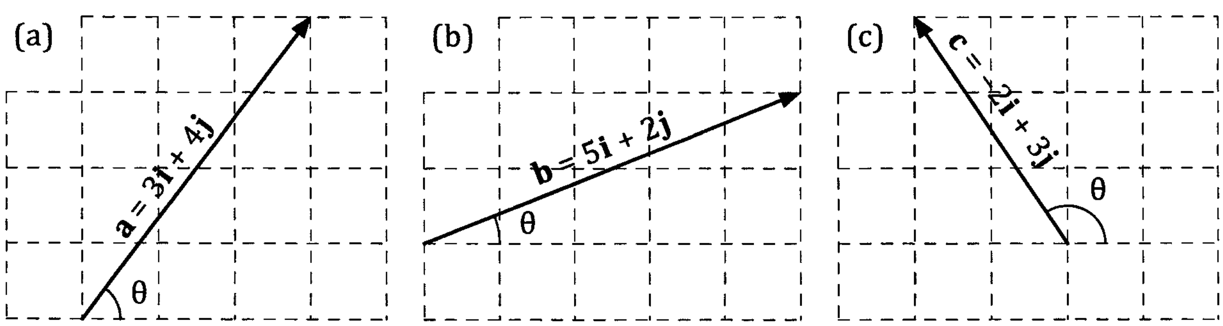
6. Calculate the magnitude of each of the vectors shown in question five.

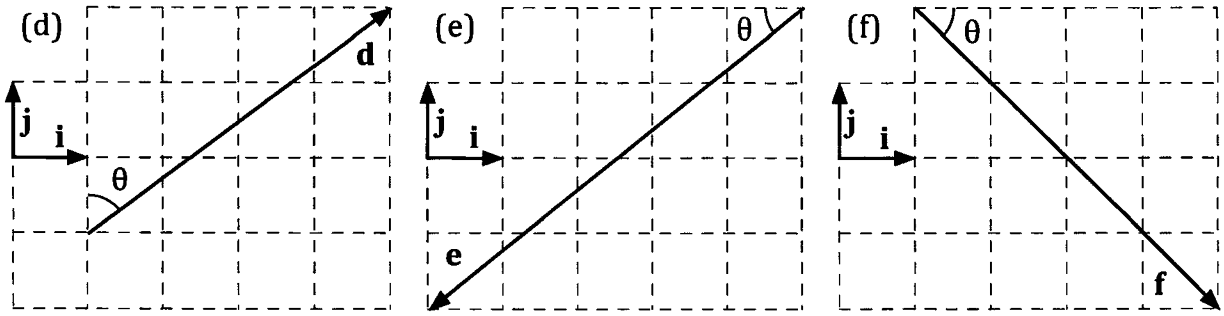
7. Calculate the magnitude of the vector $(-7\mathbf{i} + 24\mathbf{j})$ Newtons.

8. Express each of the following vectors in the form $a\mathbf{i} + b\mathbf{j}$ where $\mathbf{i}$ is a unit vector to the right and $\mathbf{j}$ is a unit vector up. (Give a and b correct to one decimal place.)



9. For each of the vectors **a** to **f** shown in this question calculate the magnitude (as an exact value) and determine the angle θ, to the nearest 0·1°.





10. An aircraft is flying at 350 km/h on a bearing 160°.
Find (a) the northerly component of the velocity,
(b) the easterly component of the velocity.

11. A vector has a westerly component of 5 units and a northerly component of 8 units.
Find the magnitude and direction of the vector.

12. If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = \mathbf{i} + 4\mathbf{j}$ find

(a) $\mathbf{a} + \mathbf{b}$,	(b) $\mathbf{a} - \mathbf{b}$,	(c) $\mathbf{b} - \mathbf{a}$,	(d) $2\mathbf{a}$,
(e) $3\mathbf{b}$,	(f) $2\mathbf{a} + 3\mathbf{b}$,	(g) $2\mathbf{a} - 3\mathbf{b}$,	(h) $-2\mathbf{a} + 3\mathbf{b}$,
(i) $\lvert\mathbf{a}\rvert$,	(j) $\lvert\mathbf{b}\rvert$,	(k) $\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert$,	(l) $\lvert\mathbf{a} + \mathbf{b}\rvert$.

13. If $\mathbf{c} = \mathbf{i} - \mathbf{j}$ and $\mathbf{d} = 2\mathbf{i} + \mathbf{j}$ find

(a) $2\mathbf{c} + \mathbf{d}$,	(b) $\mathbf{c} - \mathbf{d}$,	(c) $\mathbf{d} - \mathbf{c}$,	(d) $5\mathbf{c}$,
(e) $5\mathbf{c} + \mathbf{d}$,	(f) $5\mathbf{c} + 2\mathbf{d}$,	(g) $2\mathbf{c} + 5\mathbf{d}$,	(h) $2\mathbf{c} - \mathbf{d}$,
(i) $\lvert\mathbf{d} - 2\mathbf{c}\rvert$,	(j) $\lvert\mathbf{c}\rvert + \lvert\mathbf{d}\rvert$,	(k) $\lvert\mathbf{c} + \mathbf{d}\rvert$,	(l) $\lvert\mathbf{c} - \mathbf{d}\rvert$.

14. If $\mathbf{a} = \langle 5, 4\rangle$ and $\mathbf{b} = \langle 2, -3\rangle$ find

(a) $\mathbf{a} + \mathbf{b}$,	(b) $\mathbf{a} - \mathbf{b}$,	(c) $2\mathbf{a}$,	(d) $3\mathbf{a} + \mathbf{b}$,
(e) $2\mathbf{b} - \mathbf{a}$,	(f) $\lvert\mathbf{a}\rvert$,	(g) $\lvert\mathbf{a} + \mathbf{b}\rvert$,	(h) $\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert$.

15. If $\mathbf{c} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ find

(a) $\mathbf{c} + \mathbf{d}$,	(b) $\mathbf{c} - \mathbf{d}$,	(c) $\mathbf{d} - \mathbf{c}$,	(d) $2\mathbf{c} + \mathbf{d}$,
(e) $\mathbf{c} + 2\mathbf{d}$,	(f) $\mathbf{c} - 2\mathbf{d}$,	(g) $\lvert\mathbf{c} - 2\mathbf{d}\rvert$,	(h) $\lvert 2\mathbf{d} - \mathbf{c}\rvert$,

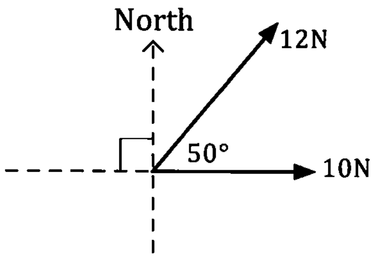
16. If $\mathbf{a} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ find the exact magnitudes of

(a) $\mathbf{a}$, (b) $\mathbf{b}$, (c) $2\mathbf{a}$, (d) $\mathbf{a} + \mathbf{b}$, (e) $\mathbf{a} - \mathbf{b}$.

17. The wing of an aircraft in flight experiences a force of 4000 N acting upwards at 20° to the vertical. Find
    (a) the magnitude of the vertical component of this force (called the lift),
    (b) the magnitude of the horizontal component of this force (the drag).

Find the resultant for each set of vectors shown in numbers 18 to 23 giving your answers in the form **ai** + b**j**, where **i** is a unit vector due east and **j** is a unit vector due north. (Give a and b correct to one decimal place).

18.



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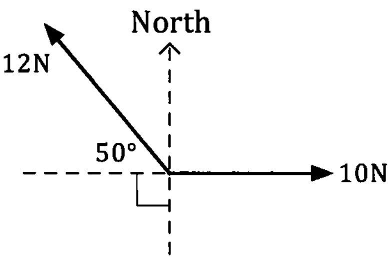

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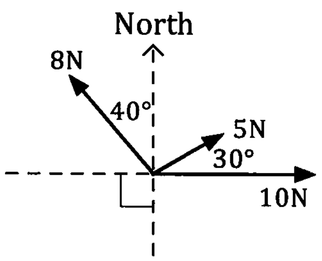

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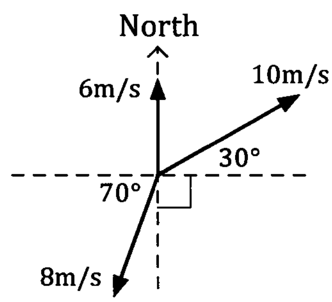

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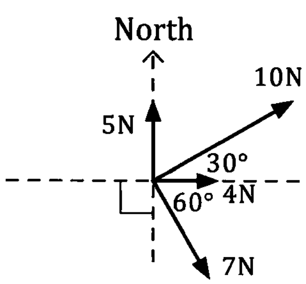

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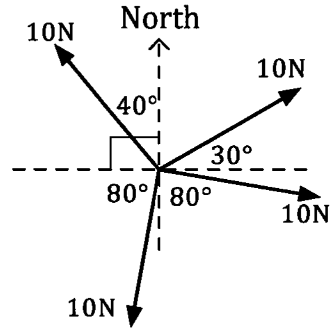


24. Forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ act on a body.
If $\mathbf{F}_1 = (2\mathbf{i} + 3\mathbf{j})$ N, $\mathbf{F}_2 = (4\mathbf{i} + 3\mathbf{j})$ N and $\mathbf{F}_3 = (2\mathbf{i} - 4\mathbf{j})$ N find the magnitude of the resultant force acting on the body.

25. Find **a** and **b** if $\mathbf{a} + \mathbf{b} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{a} - \mathbf{b} = \mathbf{i} - 7\mathbf{j}$.

26. Find **c** and **d** if $2\mathbf{c} + \mathbf{d} = -\mathbf{i} + 6\mathbf{j}$ and $2(\mathbf{c} + \mathbf{d}) = 2\mathbf{i} - 10\mathbf{j}$.

Further examples.

Example 2

If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{c} = x\mathbf{i} + \mathbf{j}$ find

(a) a vector in the same direction as **a** but twice the magnitude of **a**,

(b) a unit vector in the same direction as **a**,

(c) a vector in the same direction as **a** but the same magnitude as **b**,

(d) the possible values of x if $|\mathbf{c}| = |\mathbf{a}|$.

(a) Any vector in the same direction as **a** will be a positive scalar multiple of **a**.
To be twice the magnitude the scalar multiple must be 2.
Thus the required vector is $2\mathbf{a}$, i.e. $4\mathbf{i} + 6\mathbf{j}$.

(b) **a** has magnitude $\sqrt{2^2 + 3^2}$

$= \sqrt{13}$ units

Thus the unit vector, in the same direction as **a**, would be $\dfrac{1}{\sqrt{13}}(2\mathbf{i} + 3\mathbf{j})$

i.e. $\dfrac{2}{\sqrt{13}}\mathbf{i} + \dfrac{3}{\sqrt{13}}\mathbf{j}$

or, with rationalised denominators

$$\frac{2\sqrt{13}}{13}\mathbf{i} + \frac{3\sqrt{13}}{13}\mathbf{j}$$

unitV $\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$ $\quad\begin{bmatrix} \frac{2\cdot\sqrt{13}}{13} \\ \frac{3\cdot\sqrt{13}}{13} \end{bmatrix}$

(c) $|\mathbf{b}| = \sqrt{3^2 + (-4)^2}$

$= 5$ units.

Thus, using our answer for part (b), the vector in the same direction as **a** but of magnitude 5 units will be $\dfrac{10}{\sqrt{13}}\mathbf{i} + \dfrac{15}{\sqrt{13}}\mathbf{j}$.

(d) $|\mathbf{c}| = \sqrt{x^2 + 1^2}$ and $|\mathbf{a}| = \sqrt{2^2 + 3^2}$.

If $|\mathbf{c}| = |\mathbf{a}|$ then $x^2 + 1 = 4 + 9$

$\therefore$ $x = \pm\sqrt{12}$.

The possible values of x are $\pm 2\sqrt{3}$.

Note: A **unit vector** in the direction of vector **a** is sometimes written $\hat{\mathbf{a}}$.

Thus $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$. Similarly $\hat{\mathbf{b}} = \dfrac{\mathbf{b}}{|\mathbf{b}|}$, $\hat{\mathbf{c}} = \dfrac{\mathbf{c}}{|\mathbf{c}|}$ etc.

Example 3

A body is moving with velocity (7**i** + 24**j**) m/s. How far will it travel in twenty seconds?

If the velocity = (7**i** + 24**j**) m/s

speed = | velocity |

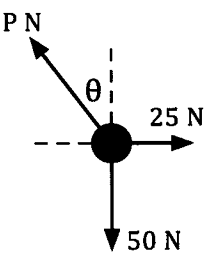
= | 7**i** + 24**j** |

= 25 m/s.

Thus in twenty seconds the body will travel 500 metres.

Example 4

The forces acting on a body are as shown in the diagram. If the body is in equilibrium find P and θ.



If the body is in equilibrium there can be no "surplus" force in any direction because, if there was, the body would tend to move in that direction.

The horizontal forces must balance. $\therefore P\sin\theta = 25$ ①

The vertical forces must balance. $\therefore P\cos\theta = 50$ ②

Dividing ① by ② gives $\tan\theta = 0{\cdot}5$

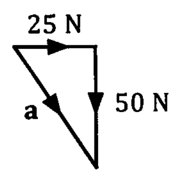
$\theta \approx 26{\cdot}6°$

Substituting for θ into ① gives $P \approx 55{\cdot}9$

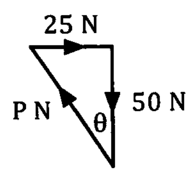
If the body is in equilibrium then P ≈ 56 and θ ≈ 27°.

Alternatively we could approach example 4 as follows:

The resultant of the 25 N force and the 50 N force is a vector **a**, see diagram on the right.



Thus the force of P N must be −**a** to counteract the effect of **a** and reduce the system to equilibrium.



By Pythagoras: $P = \sqrt{25^2 + 50^2}$

i.e. $P \approx 56$

By trigonometry: $\tan\theta = \frac{25}{50}$

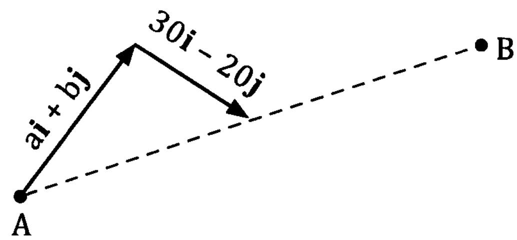
i.e. $\theta \approx 27°$

If the body is in equilibrium then P ≈ 56 and θ ≈ 27°, as before.

Example 5

Airports A and B are such that $\overrightarrow{AB}$ = (600**i** + 200**j**) km. An aircraft is to be flown directly from A to B. The aircraft can maintain a steady speed of 390 km/h in still air. There is a wind blowing with velocity (30**i** - 20**j**) km/h.
Find, in the form **ai** + **bj**, the velocity vector the pilot should set so that this velocity, together with the wind, causes the plane to travel directly from A to B?

The resultant of the planes velocity due to its engines, (**ai** + **bj**) km/h, and the wind, (30**i** - 20**j**) km/h, must be along $\overrightarrow{AB}$. i.e. the resultant must be a positive scalar multiple of 600**i** + 200**j**.



Thus $\quad a\mathbf{i} + b\mathbf{j} + 30\mathbf{i} - 20\mathbf{j} = \lambda(600\mathbf{i} + 200\mathbf{j}) \qquad \lambda > 0$

Equating the **i** components $\quad a + 30 = 600\lambda$ ①

Equating the **j** components $\quad b - 20 = 200\lambda$ ②

Dividing ① by ② $\quad \dfrac{a+30}{b-20} = \dfrac{600}{200}$

i.e. $\quad a = 3b - 90$ ③

We also know that $\quad |a\mathbf{i} + b\mathbf{j}| = 390 \qquad$ i.e. $a^2 + b^2 = 390^2$ ④

From ③ and ④ $\quad (3b - 90)^2 + b^2 = 390^2$

Using a calculator to solve this equation gives $\quad b = 150$ or $b = -96$

But λ must be positive so, from ②, b ≠ −96.

(Note: b = −96 gives a resultant in direction $\overrightarrow{BA}$, not $\overrightarrow{AB}$ as required.)

If $\quad b = 150, \quad a = 360 \qquad$ i.e. velocity = 360**i** + 150**j**.

The required velocity is (360**i** + 150**j**) km/h.

Note
- The above example could be solved by instead changing the given vectors to magnitude and direction form and then solving as in chapter 3.
- To avoid some of the algebra we could use the ability of some calculators to solve equations ①, ② and ④ simultaneously to determine a, b and λ.
- The justification for "equating the **i** components" and "equating the **j** components" is given below:

  If $\quad a\mathbf{i} + b\mathbf{j} = c\mathbf{i} + d\mathbf{j} \qquad$ then $\quad a\mathbf{i} - c\mathbf{i} = d\mathbf{j} - b\mathbf{j}$

  i.e. $\quad (a - c)\mathbf{i} = (d - b)\mathbf{j}$

  As **i** and **j** are not parallel it follows that $\quad a - c = 0$ and $d - b = 0$.

  i.e. $\quad a = c$ and $d = b$.

We usually choose to express vectors in terms of the horizontal unit vector, **i**, and vertical unit vector, **j**, because these are convenient directions. However any two non-parallel vectors can be chosen as base vectors if horizontal and vertical are not convenient.

Example 6

Using **a** = 2**i** + 3**j** and **b** = 4**i** – **j** as base vectors express each of the following in the form $\lambda\mathbf{a} + \mu\mathbf{b}$ (a) 5**i** + 3**j**, (b) 6**i** – 4**j**.

(a) Let $5\mathbf{i} + 3\mathbf{j} = \lambda\mathbf{a} + \mu\mathbf{b}$

i.e. $5\mathbf{i} + 3\mathbf{j} = \lambda(2\mathbf{i} + 3\mathbf{j}) + \mu(4\mathbf{i} - \mathbf{j})$

Equating the **i** components $5 = 2\lambda + 4\mu$ ①

Equating the **j** components $3 = 3\lambda - \mu$ ②

Solving ① and ② simultaneously: $\lambda = \frac{17}{14}$ and $\mu = \frac{9}{14}$

Thus $5\mathbf{i} + 3\mathbf{j} = \frac{17}{14}\mathbf{a} + \frac{9}{14}\mathbf{b}$

(b) Let $6\mathbf{i} - 4\mathbf{j} = \lambda\mathbf{a} + \mu\mathbf{b}$

i.e. $6\mathbf{i} - 4\mathbf{j} = \lambda(2\mathbf{i} + 3\mathbf{j}) + \mu(4\mathbf{i} - \mathbf{j})$

Equating the **i** components $6 = 2\lambda + 4\mu$ ①

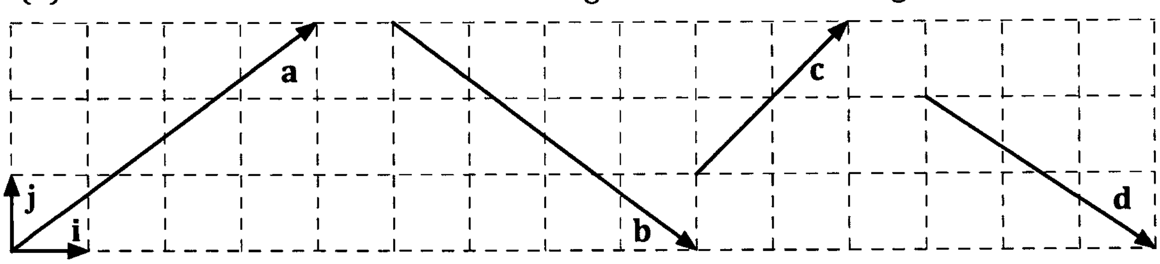
Equating the **j** components $-4 = 3\lambda - \mu$ ②

Solving simultaneously $\lambda = -\frac{5}{7}$ and $\mu = \frac{13}{7}$

Thus $6\mathbf{i} - 4\mathbf{j} = -\frac{5}{7}\mathbf{a} + \frac{13}{7}\mathbf{b}$

Exercise 4B

1. For <u>each</u> of the vectors **a** to **d** shown below, write in the form h**i** + k**j** ....
   (a) the vector itself,
   (b) a vector in the same direction as the given vector but twice as long,
   (c) a unit vector in the same direction as the given vector,
   (d) a vector in the same direction as the given vector but of length 2 units.



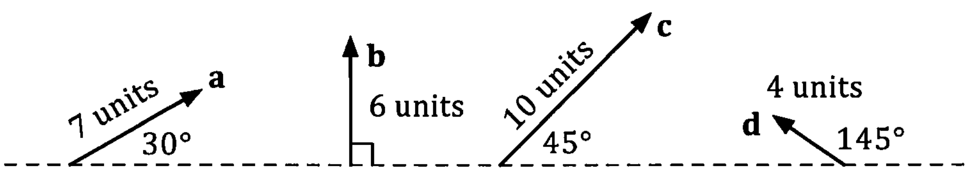
2. Given that $\mathbf{a} = -3\mathbf{i} + 4\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{c} = 3\mathbf{i} - 2\mathbf{j}$ find
   (a) a unit vector in the same direction as **b**,
   (b) a vector in the same direction as **b** but equal in magnitude to **a**,
   (c) a vector in the same direction as **a** but equal in magnitude to **c**,
   (d) a vector in the same direction as the resultant of **a**, **b** and **c** but equal in magnitude to **a**.

3. For this question $\mathbf{a} = 2\mathbf{i} - 4\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} + 2\mathbf{j}$, $\mathbf{c} = \mathbf{i} - 8\mathbf{j}$, $\mathbf{d} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{e} = 4\mathbf{i} - 2\mathbf{j}$.
   (a) Which of these vectors are parallel to each other?
   (b) Find the resultant of **a**, **b**, **c**, **d** and **e**.
   (c) Find the magnitude of this resultant.
   (d) Taking the direction of the unit vector **j** as due North and **i** as due East express the direction of the resultant as a bearing, measured clockwise from North and to the nearest degree.

4. For this question $\mathbf{a} = w\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = -\mathbf{i} + x\mathbf{j}$, $\mathbf{c} = 0{\cdot}5\mathbf{i} + y\mathbf{j}$ and $\mathbf{d} = -\mathbf{i} - z\mathbf{j}$.
   Find the possible values of w, x, y and z given that all of the following are true:
   - **a** is of magnitude 5 units and w is negative,
   - **b** is parallel to **a**,
   - **c** is a unit vector,
   - the resultant of **a** and **d** has magnitude 13 units.

5. For this question $\mathbf{p} = 0{\cdot}6\mathbf{i} - a\mathbf{j}$, $\mathbf{q} = b\mathbf{i} + c\mathbf{j}$, $\mathbf{r} = d\mathbf{i} + e\mathbf{j}$ and $\mathbf{s} = f\mathbf{i} + g\mathbf{j}$.
   Find the possible values of a, b, c, d, e, f and g given that all of the following are true:
   - **p** is a unit vector and a is a positive constant,
   - **q** is in the same direction as **p** and five times the magnitude,
   - $\mathbf{r} + 2\mathbf{q} = 11\mathbf{i} - 20\mathbf{j}$,
   - **s** is in the same direction as **r** but equal in magnitude to **q**.

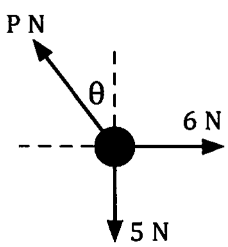
6. Find, correct to one decimal place, the magnitude of **R** given that **R** is the resultant of **a**, **b**, **c** and **d** shown below.



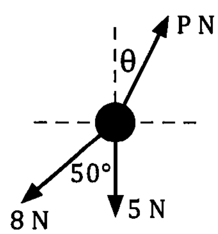
   Express in the form $\lambda\mathbf{i} + \mu\mathbf{j}$ the vector **e** such that $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + \mathbf{e} = 0$. (Give λ and μ correct to one decimal place.)

Find P, correct to one decimal place, and θ, to the nearest degree, in each of the following if in each case the body is in equilibrium.

7.



8.



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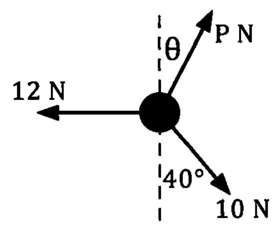
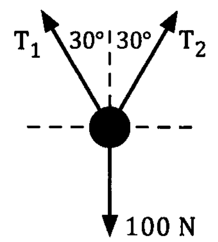
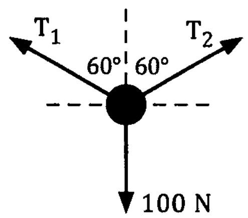
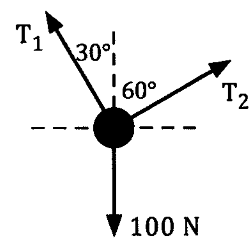


Each of the following diagrams shows a body of weight 100 N, supported in equilibrium by two wires. Find the magnitudes of T_1 and T_2, the tensions in the wires.

10. 

11. 

12. 

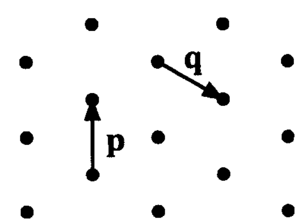
13. Two particles, A and B, have velocities of (21**i** + 17**j**) m/s and (26**i** - 2**j**) m/s respectively. Which particle is moving the fastest?

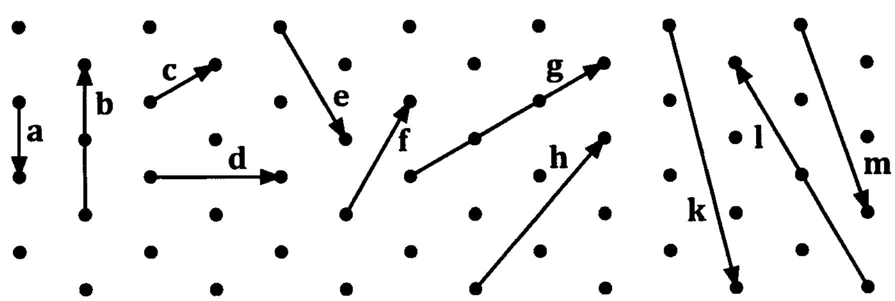
14. How far will a body moving with a constant velocity of (5**i** - 2**j**) m/s travel in one minute?

15. A helicopter can fly at 75m/s in still air. The pilot wishes to fly from airport A to a second airport B, 300 km due North of A. If **i** is a unit vector due East and **j** a unit vector due North find (in the form a**i** + b**j**) the velocity vector that the pilot should set and the time the journey will take if
    (a) there is no wind blowing,
    (b) there is a wind of (21**i** + 10**j**) m/s blowing.

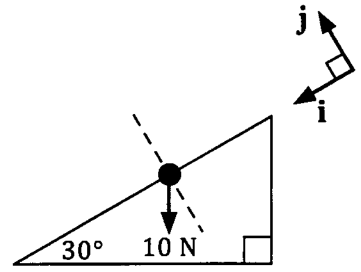
16. The helicopter of question 15 now wishes to return from B to A.
    If the wind of (21**i** + 10**j**) m/s still blows what velocity vector should the pilot now set and how long will the journey take?

17. An engineer finds that for a lot of her work she uses isometric paper. For her the usual base vectors →**i** and ↑**j** are not so useful. Instead she expresses vectors in terms of two base vectors that she calls **p** and **q** (see the diagram on the right).
    Express each of the vectors shown below in terms of **p** and/or **q**.





18. The diagram shows a particle of weight 10 N on an inclined plane that is angled at 30° to the horizontal. Express this weight as a vector (a**i** + b**j**) N where **i** and **j** are unit vectors down and perpendicular to the slope respectively (see diagram).



19. An unconventional mathematician decides not to use the horizontal and vertical unit vectors **i** and **j** as base vectors but instead wishes to express all vectors in terms of new base vectors **a** and **b**, where:

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} \qquad \text{and} \qquad \mathbf{b} = \mathbf{i} - \mathbf{j}.$$

Express each of the following in the form $x\mathbf{a} + y\mathbf{b}$.

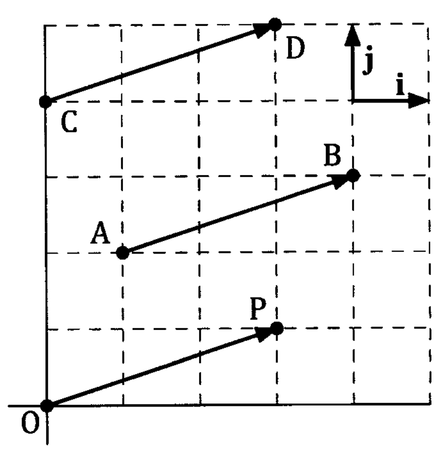
(a)	$3\mathbf{i} + 2\mathbf{j}$	(b)	$5\mathbf{i} + 5\mathbf{j}$	(c)	$\mathbf{i} + 9\mathbf{j}$
(d)	$4\mathbf{i} + 7\mathbf{j}$	(e)	$3\mathbf{i} - \mathbf{j}$	(f)	$3\mathbf{i} + 7\mathbf{j}$

20. Airports A and B are such that $\overrightarrow{AB} = (-250\mathbf{i} + 750\mathbf{j})$ km.
An aircraft is to be flown directly from A to B. In still air the aircraft can maintain a steady speed of 400 km/h. There is a wind blowing with velocity $(-13\mathbf{i} - 9\mathbf{j})$ km/h. Find, in the form a**i** + b**j**, the velocity vector the pilot should set so that this velocity, together with the wind, causes the plane to travel directly from A to B.
If the wind remains unchanged find, in the form a**i** + b**j**, the velocity vector the pilot should now set to return directly from B to A.

Position vectors.

Consider the points O, A, B, C, D and P shown in the diagram.

The vectors $\overrightarrow{OP}$, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are each $3\mathbf{i} + \mathbf{j}$.
However, with O as the origin, only point P has a **position vector** of $3\mathbf{i} + \mathbf{j}$.

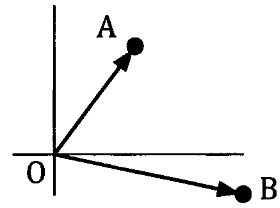


Point A has position vector $\mathbf{i} + 2\mathbf{j}$,
Point B has position vector $4\mathbf{i} + 3\mathbf{j}$,
Point C has position vector $4\mathbf{j}$,
Point D has position vector $3\mathbf{i} + 5\mathbf{j}$.

Example 7

Points A and B have position vectors $2\mathbf{i} + 3\mathbf{j}$ and $5\mathbf{i} - \mathbf{j}$ respectively. Find $\overrightarrow{AB}$.

Initially draw a rough sketch of the situation:



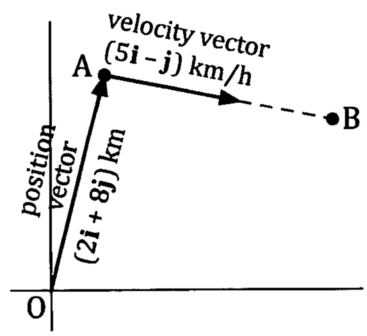
From the diagram
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -\overrightarrow{OA} + \overrightarrow{OB}\\ &= -(2\mathbf{i} + 3\mathbf{j}) + (5\mathbf{i} - \mathbf{j})\\ &= 3\mathbf{i} - 4\mathbf{j}\end{aligned}$$

Thus $\overrightarrow{AB} = 3\mathbf{i} - 4\mathbf{j}$.

Example 8

At 1 p.m. a ship is at a location A, position vector $(2\mathbf{i} + 8\mathbf{j})$ km, and is moving with velocity $(5\mathbf{i} - \mathbf{j})$ km/h. If the ship continues with this velocity what will be its position vector at 4 p.m.?

Suppose the ship is at point B at 4 p.m. (see diagram).



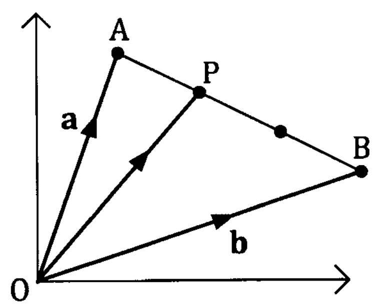
Then
$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB}\\ &= (2\mathbf{i} + 8\mathbf{j}) + 3(5\mathbf{i} - \mathbf{j})\\ &= 17\mathbf{i} + 5\mathbf{j}\end{aligned}$$

By 4 p.m. the ship will be at the point with position vector $(17\mathbf{i} + 5\mathbf{j})$.

Example 9

Points A and B have position vectors $\mathbf{i} + 7\mathbf{j}$ and $10\mathbf{i} + 4\mathbf{j}$ respectively. Find the position vector of the point that divides AB internally in the ratio 1 : 2.
(Note: If a point P divides AB in the ratio 1 : 2 then AP : PB = 1 : 2.)

We require the position vector of the point P (see diagram) where AP : PB = 1 : 2



$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP}\\ &= \overrightarrow{OA} + \tfrac{1}{3}\overrightarrow{AB}\\ &= \mathbf{a} + \tfrac{1}{3}(-\mathbf{a} + \mathbf{b})\\ &= \tfrac{2}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{b}\\ &= \tfrac{2}{3}(\mathbf{i} + 7\mathbf{j}) + \tfrac{1}{3}(10\mathbf{i} + 4\mathbf{j})\\ &= 4\mathbf{i} + 6\mathbf{j}\end{aligned}$$

The point dividing AB internally in the ratio 1 : 2 has position vector $4\mathbf{i} + 6\mathbf{j}$.

Exercise 4C

1. Points A, B, C and D have cartesian coordinates (2, 5), (−3, 6), (0, −5) and (3, 8) respectively. Give the position vectors of each point in the form $a\mathbf{i} + b\mathbf{j}$ where $\mathbf{i}$ is a unit vector in the direction of the positive x-axis and $\mathbf{j}$ is a unit vector in the direction of the positive y-axis.

2. Points A and B have position vectors $3\mathbf{i} + \mathbf{j}$ and $2\mathbf{i} - \mathbf{j}$ respectively.

   Express (a) $\overrightarrow{AB}$ and (b) $\overrightarrow{BA}$ in the form $a\mathbf{i} + b\mathbf{j}$.

3. Points A, B and C have position vectors $-\mathbf{i} + 4\mathbf{j}$, $2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{i} + 5\mathbf{j}$ respectively.

   Express (a) $\overrightarrow{AB}$, (b) $\overrightarrow{BC}$ and (c) $\overrightarrow{CA}$ in the form $a\mathbf{i} + b\mathbf{j}$.

4. Points A, B, C and D have position vectors $\mathbf{i} + 2\mathbf{j}$, $4\mathbf{i} - 2\mathbf{j}$, $-\mathbf{i} + 11\mathbf{j}$ and $6\mathbf{i} - 13\mathbf{j}$ respectively. Find (a) $\overrightarrow{AB}$, (b) $\overrightarrow{BC}$, (c) $\overrightarrow{CD}$, (d) a vector in the same direction as $\overrightarrow{AB}$ but equal in magnitude to $\overrightarrow{CD}$.

5. With respect to an origin O, points A and B have position vectors $3\mathbf{i} + 7\mathbf{j}$ and $-2\mathbf{i} + \mathbf{j}$ respectively. Find (a) $|\overrightarrow{OA}|$, (b) $|\overrightarrow{OB}|$, (c) $|\overrightarrow{AB}|$.

6. Points A, B and C have position vectors $2\mathbf{i} + 3\mathbf{j}$, $5\mathbf{i} - \mathbf{j}$ and $3\mathbf{i} + 7\mathbf{j}$ respectively.

   Find (a) $|\overrightarrow{AB}|$, (b) $|\overrightarrow{BA}|$, (c) $|\overrightarrow{AC}|$, (d) $|\overrightarrow{BC}|$.

7. Points A and B have position vectors $-\mathbf{i} + 6\mathbf{j}$ and $5\mathbf{i} + 3\mathbf{j}$ respectively.
   How far is (a) A from the origin, (b) B from the origin, (c) A from B?

8. With respect to an origin O, points A, B, C and D have position vectors
   $2\mathbf{i} - 3\mathbf{j}$, $\mathbf{i} + 2\mathbf{j}$, $9\mathbf{i} + 21\mathbf{j}$ and $6\mathbf{i} - 2\mathbf{j}$ respectively.

   Find (a) $\overrightarrow{AB}$ in component form, (b) $\overrightarrow{BC}$ in component form,

   (c) $\overrightarrow{CD}$ in component form, (d) $|\overrightarrow{AC}|$,

   (e) $\overrightarrow{OA} + \overrightarrow{AB}$ in component form, (f) $\overrightarrow{OA} + 2\overrightarrow{AC}$ in component form.

9. If point A has position vector $3\mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{AB} = 7\mathbf{i} - \mathbf{j}$ find the position vector of point B.

10. Point A has position vector $-\mathbf{i} + 7\mathbf{j}$, $\overrightarrow{AB} = 2\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{AC} = 4\mathbf{i} - 3\mathbf{j}$.
    Find (a) the position vector of point B,
    (b) the position vector of point C,
    (c) $\overrightarrow{BC}$.

11. Point A has position vector $-\mathbf{i} + 9\mathbf{j}$. Point C has position vector $7\mathbf{i} - \mathbf{j}$. Points B and D are such that $\overrightarrow{BC} = 4\mathbf{i} - 6\mathbf{j}$ and $\overrightarrow{DC} = 3\mathbf{i} + 2\mathbf{j}$.
    Find (a) the position vector of point B, (b) the position vector of point D,
    (c) $\overrightarrow{BD}$, (d) $|\overrightarrow{AD}|$.

12. A particle has an initial position vector of $(2\mathbf{i} + 9\mathbf{j})$ m with respect to an origin O. If the particle moves with a constant velocity of $(2\mathbf{i} - 5\mathbf{j})$ m/s what will be its position vector after
    (a) 1 second, (b) 2 seconds, (c) 10 seconds?
    (d) How far is the particle from the origin after five seconds?

13. A particle has an initial position vector of $(5\mathbf{i} - 6\mathbf{j})$ m with respect to an origin O. If the particle moves with a constant velocity of $(\mathbf{i} + 6\mathbf{j})$ m/s what will be its position vector after
    (a) 2 seconds, (b) 3 seconds, (c) 7 seconds?
    (d) How far is the particle from the origin after five seconds?
    (e) After how many seconds will the particle be 50 metres from the origin?

14. Points A, B and C have position vectors $3\mathbf{i} - \mathbf{j}$, $-\mathbf{i} + 15\mathbf{j}$ and $9\mathbf{i} - 25\mathbf{j}$ respectively. Use vectors to prove that A, B and C are collinear.

15. Points D, E and F have position vectors $9\mathbf{i} - 7\mathbf{j}$, $-11\mathbf{i} + 8\mathbf{j}$ and $25\mathbf{i} - 19\mathbf{j}$ respectively. Use vectors to prove that D, E and F are collinear.

16. Points A and B have position vectors $2\mathbf{i} + 5\mathbf{j}$ and $12\mathbf{i} + 10\mathbf{j}$ respectively. Find the position vector of the point dividing AB internally in the ratio 4 : 1.

17. Points A and B have position vectors $-2\mathbf{i} + 2\mathbf{j}$ and $10\mathbf{i} - \mathbf{j}$ respectively. Find the position vector of the point that divides AB internally in the ratio 1 : 2.

18. Points A and B have position vectors $\mathbf{i} + 8\mathbf{j}$ and $19\mathbf{i} + 2\mathbf{j}$ respectively. Find the position vector of the point that divides AB internally in the ratio 1 : 4.

Miscellaneous Exercise Four.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

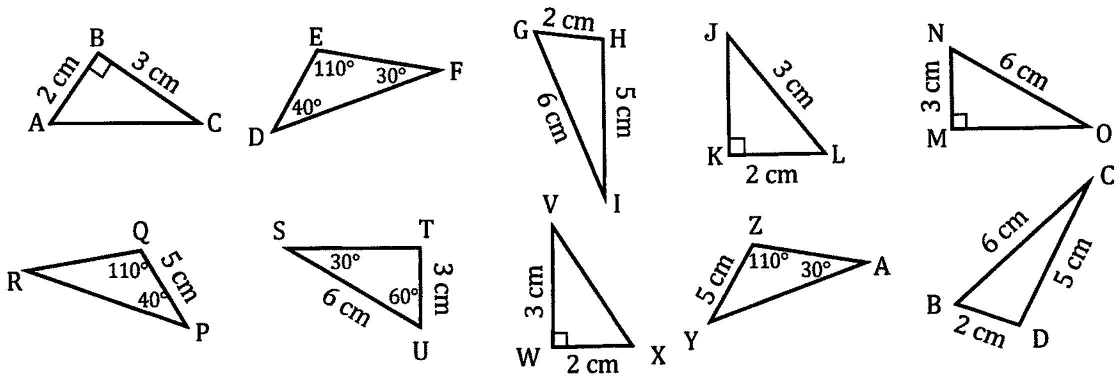
1. If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j}$ find λ and μ such that $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$.

2. Two or three letter codes are to be formed using the letters A, B, C, D, E, F.
Each code must not use the same letter more than once.
How many such codes are possible?

3. If a positive whole number ends in a five then the number is a multiple of five.
Write the converse statement and the contrapositive statement and in each case state whether true or false.

4. A careers awareness form requires a student to choose five careers from a list of twelve and write them in order of preference, putting most preferred first to least preferred last. How many different ordered lists are possible?

5. From the triangles shown below find four pairs that must be congruent, stating the reason each time. (Diagrams not necessarily drawn to scale.)



6. Points A and B have position vectors $a\mathbf{i} - 15\mathbf{j}$ and $10\mathbf{i} + b\mathbf{j}$ respectively.
The point C, position vector $4\mathbf{i} - 3\mathbf{j}$, lies on AB, between A and B, and is such that
$$AC : CB = 2 : 3.$$
Find the values of a and b.

7. How many four or five digit numbers greater than 5000 can be made using some or all of the digits 1, 2, 3, 4 and 5 if
(a) a digit cannot appear more than once in a number,
(b) a digit can appear more than once in a number.

8. In a cricket match the batsman is at a point O and he hits the ball with a velocity of $(7\mathbf{i} + 24\mathbf{j})$ m/s. A fielder at point A, position vector $2\mathbf{i} + 3\mathbf{j}$ relative to O, does not move at all as the ball passes by. Assuming the ball suffers no change in velocity:
(a) How long will the ball take to reach the boundary, 60 metres from O?
(b) Find the least distance between the fielder at point A and the ball during its journey to the boundary.